Two "almost sure" theorems under the lens of algorithmic randomness

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K-trivials, Martin-Löf randoms, and $\leq_T$

- $A \subseteq \{0,1\}^{\mathbb{N}}$ is called K-trivial if $\forall n K(A \upharpoonright n) \leq K(n) + O(1)$.
- If $Z \not\geq_T \emptyset'$ is MLR, A is c.e. and $A \leq_T Z$, then A is K-trivial [HNS, 2007]. Does every K-trivial A have such a Z Turing above?

The following two results together show the answer is YES, in fact via a single Z.

Theorem (BGKNT, JEMS 2016)

Let Z be MLR. Then

Z is Oberwolfach random $\iff$

Z does **not** compute every K-trivial.

Theorem (by Day and Miller, MRL, 2015)

There is $Z \not\geq_T \emptyset'$ that is MLR and not Oberwolfach random.

Interactions of Algorithmic Randomness

Randomness

- Defines hierarchy of rdness notions via algorithmic tests
- Provides characterizations
- Separations

Computability

- **Complexity** of subsets of N
- Relative via Turing reducibility
- Absolute, such as low, below halting problem, ...

Analysis, ergodic theory

"Almost sure" theorems

Algorithmic test notions

Work in $[0, 1]$ or in $\{0, 1\}^{\mathbb{N}}$. Most test notions refine this:
A (weak 2) test is a uniformly Σ^0_1 sequence $(G_m)_{m \in \mathbb{N}}$ such that $\lim_m \lambda G_m = 0$. Z fails the test if $Z \in \bigcap G_m$.

- OW test (or left-c.e. bounded test): $\lambda G_m = O(\beta - \beta_m)$, where (β_m) is a computable sequence of rationals, and $\beta = \sup_m \beta_m$.
- Martin-Löf test: β is computable (equivalently, could require $\lambda G_m \leq 2^{-m}$)
- Schnorr test: λG_m is computable uniformly in m.

Interactions Computability-Randomness

- LOWNESS PROPS
- K-TRIVIALS
- $\le_{LR}$, $\le_{ML}$
- TURING DEGREES
- MUCHNIK DEGREES

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Interactions Randomness- Analysis

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ALGORITHMIC VERSIONS
OF ALMOST-SURE THEOREMS

HOW MUCH RANDOMNESS
IS NEEDED FOR THE
PROPERTY TO HOLD?

Analysis, ergodic theory

"Almost sure" theorems

Canad. Math. Bull. Vol. **21** (4), 1978

TRANSLATES OF SEQUENCES IN SETS OF POSITIVE MEASURE

BY

D. BORWEIN AND S. Z. DITOR

Given a measurable set A of real numbers with measure $mA > 0$, and a sequence $\{d_n\}$ of real numbers converging to zero, is there always an x such that $x + d_n \in A$ for all n sufficiently large?

The answer to this question, which was posed to the authors by P. Erdös, is NO. The actual situation can be described as follows.

THEOREM 1. (i) *If A is a measurable set with $mA > 0$ and $\{d_n\}$ is a sequence converging to zero, then, for almost all $x \in A$, $x + d_n \in A$ for infinitely many n.*

(ii) *There is a measurable set A with $mA > 0$ and a monotonic sequence $\{d_n\}$ of positive numbers converging to zero such that, for all x, $x + d_n \notin A$ for infinitely many n.*

Borwein-Ditor Theorem

Theorem (Borwein-Ditor, 1978)

Let $A \subseteq \mathbb{R}$ be closed. Let (r_m) be a null sequence in $\mathbb{R}$.
Almost surely,

$$x \in A \Rightarrow x \in A + r_m \text{ for infinitely many } m.$$

- The proof is easy: The set $E = \{x \colon \exists^\infty m \, [x \in A + r_m]\}$ is contained in A because A is closed. Also $\lambda E \geq \lambda A$.
- Result fails with "for a.e. m": Borwein and Ditor build an A and null sequence so that for each $x \in \mathbb{R}$, also $\exists^\infty m \, x \notin A + r_m$. All computable.
- If A is effectively closed and (r_m) is computable, then each 1-generic x does this. So we don't get a test notion out of this.

Theorem (Galicki and N., Proceedings of CiE 2016)

Let $\mathcal{P} \subseteq \mathbb{R}$ be effectively closed, (r_m) computable null sequence.

(a) If x is OW-random then $\exists m \, [x \in \mathcal{P} + r_m]$ (hence $\exists^\infty$).

(b) In case $\lambda\mathcal{P}$ is computable, Schnorr randomness suffices.

Sketch: Work in $\{0,1\}^{\mathbb{N}}$ instead of $[0,1]$ (i.e., ignore dyadic rationals).
$\mathcal{S} := \{0,1\}^{\mathbb{N}} - \mathcal{P} = \bigcup_m [\sigma_m]$ for a computable sequence of strings (σ_m).
Can assume $r_m \leq 2^{-m}$. Let q be a computable function with

$$q(m) > m \text{ and } \forall i < m \, [q(m) > |\sigma_i|].$$

A test that x fails is obtained by shifting $\mathcal{S}$ by $r_{q(m)}$ and removing the first m dyadic intervals unshifted:

$$G_m = (\mathcal{S} + r_{q(m)}) \setminus [\sigma_0, \ldots, \sigma_{m-1}]^{\prec}.$$

(a) (G_m) is OW test via $\beta = \lambda\mathcal{S}$, $\beta_m = \lambda[\sigma_0, \ldots, \sigma_{m-1}]^{\prec} - m2^{-m}$.
(b) β is computable by hypothesis, so (G_m) is Schnorr test.

Any left-c.e. ML-random fails the BD property

We say that Z has the BD property if for each $\mathcal{P} \subseteq \mathbb{R}$ be effectively closed, (r_m) computable null sequence, $Z \in \mathcal{P}$ implies that $Z \in \mathcal{P} + r_m$ for some m.

Let Z be a left-c.e. ML random.

- By the usual existence of a universal ML test, $Z = \min(\mathcal{P})$ for some effectively closed set of ML-randoms.
- Thus Z fails the BD property via any computable null sequence of negative real numbers.

We don't know of substantially different examples of Turing complete ML-randoms that fail BD.
In fact each ML-random with the BD property could be Turing incomplete.

Version for k null sequences

Theorem (Borwein-Ditor, 1978)

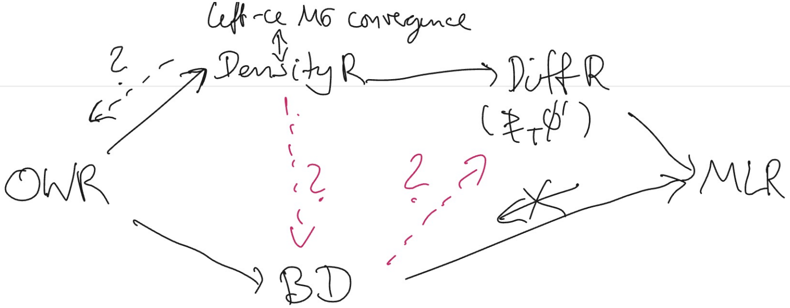
Let $A \subseteq \mathbb{R}$ be closed. Let $k \in \mathbb{N}$. For each $\ell < k$ let (r^ℓ_m) be a null sequence in $\mathbb{R}$. Then almost surely

$$x \in A \Rightarrow \forall \ell < k\, [x \in A + r^\ell_m] \text{ for infinitely many } m.$$

Their easy proof using that A is closed doesn't work. However, in the algorithmic settings, the proof goes through almost unchanged.

Comparison with density randomness

- For $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ and $Z \in \{0,1\}^{\mathbb{N}}$ one defines the lower density
$$\underline{\rho}(\mathcal{P} \mid Z) = \liminf_k \lambda(\mathcal{P} \cap [Z \upharpoonright k])/2^{-k}.$$
- We say that $Z \in \mathsf{MLR}$ is density random if $\underline{\rho}(\mathcal{P} \mid Z) = 1$ for each effectively closed $\mathcal{P}$ with $Z \in \mathcal{P}$. (See Miyabe N and Zhang, BSL, 2016)



Multiple recurrence

We work mainly in the setting $\{0,1\}^{\mathbb{N}}$ with the shift operator $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ that erases the first bit. Note that T is measure preserving (ergodic in fact).

Definition

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be measurable, and let $Z \in \{0,1\}^{\mathbb{N}}$. We say that Z is k-recurrent in $\mathcal{P}$ if there is $n \geq 1$ such that

$$\forall i. 1 \leq i \leq k\ [T^{ni}(Z) \in \mathcal{P}].$$

We say that Z is multiply recurrent in $\mathcal{P}$ if Z is k-recurrent in $\mathcal{P}$ for each $k \geq 1$.

By a general result of Furstenberg, if $\lambda\mathcal{P} > 0$, then almost every Z is multiply recurrent in $\mathcal{P}$.

Algorithmic versions

Theorem (Downey, Nandakumar, N. 2019)

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be effectively closed with $0 < \alpha := \lambda\mathcal{P}$.

(a) Each Martin-Löf random Z is multiply recurrent in $\mathcal{P}$.

(b) If α is computable then Schnorr randomness suffices.

(c) If $\mathcal{P}$ is clopen then Kurtz randomness suffices.

Idea for (a): Fix k, and assume Z is not k-recurrent.
First assume that $1 - \alpha < 1/k$. We can build a ML-test (G_m) for Z that looks for failures of k-recurrence on longer and longer initial segments. By the hypothesis, the measure of such a failure is at most $q = k(1-\alpha) < 1$. Then iterate this m times for G_m with $\lambda G_m < q^m$.
To remove the additional hypothesis, write $\{0,1\}^{\mathbb{N}} - \mathcal{P} = \bigcup_r [\tau_r]$ for a computable prefix free sequence, and work with $\mathcal{P} \cup \bigcup_{i<r} [\tau_i]$ instead where r is large enough so that hypothesis holds. Extra complications.

Recurrence for k shift operators

The probability space under consideration is now $\mathcal{X} = \{0,1\}^{\mathbb{N}^k}$ with the product measure. For $1 \le i \le k$

$$T_i(Z)(u_1, \dots, u_k) = Z(u_1, \dots, u_i + 1, \dots, u_k).$$

Z is recurrent in a class $\mathcal{P} \subseteq \mathcal{X}$ if $[Z \in \bigcap_{i \le k} T_i^{-n}(\mathcal{P})]$ for some n.

Theorem

Let $\mathcal{P} \subseteq \mathcal{X}$ be a Π^0_1 class with $0 < p = \lambda\mathcal{P}$. Let $Z \in \mathcal{X}$.
If Z is (a) ML-random (b) Schnorr random (c) Kurtz-random, then Z is recurrent in $\mathcal{P}$ assuming also that for (b) $\lambda\mathcal{P}$ is computable, for (c) $\mathcal{P}$ is clopen.

What is the full result?

We don't have an algorithmic version of the general multiple recurrence theorem.

Conjecture

- Let (X, μ) be a computable probability space.
- Let $T_1, \ldots, T_k$ be computable measure preserving transformations that commute pairwise.
- Let $\mathcal{P}$ be effectively closed with $\mu P > 0$.

If $z \in \mathcal{P}$ is ML-random then $\exists n[z \in \bigcap_{i \leq k} T_i^{-n}(\mathcal{P})]$.

Some references

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